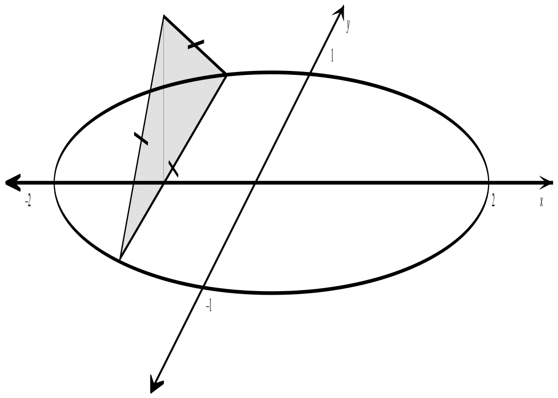
Ext 2 Task 3 T2 2014

QUESTION 1 (15 Marks) Marks

(a) A solid shape has as its base an ellipse in the *XY* plane as shown below. Sections taken perpendicular to the *X*-axis are equilateral triangles. The major and minor axes of the ellipse are 4 metres and 2 metres respectively.



(i) Write down the equation of the ellipse. 1

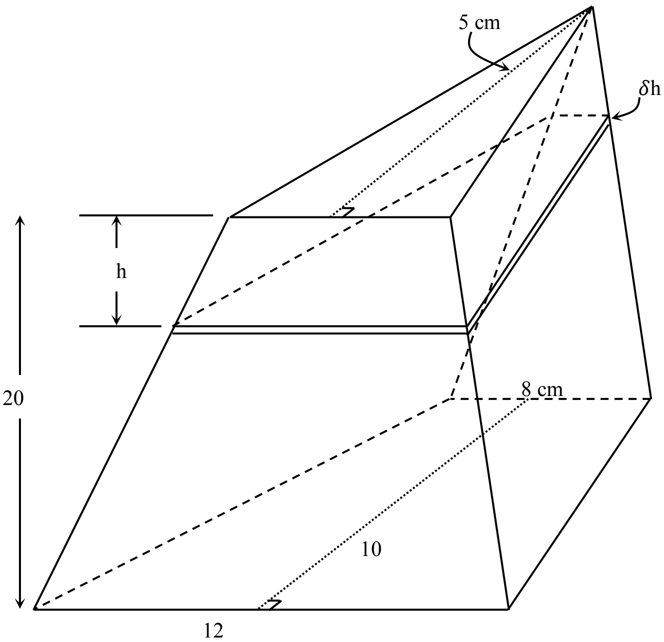
(ii) Show that the area of the cross-section at $x = k$ is given by 2

$$A = \frac{\sqrt{3}}{4}\left(4 - k^2\right).$$

(iii) By using the technique of slicing, find the volume of the solid. 2

(b)

The solid shown has a base which is a trapezium. The parallel sides are 12cm and 8cm. The perpendicular height is 10cm. Each slice taken parallel to the base is a trapezium with one of the parallel sides 4cm shorter than the other side. The top of the figure (which is parallel to the base) is a triangle with a height of 5cm. The height of the solid is 20cm.



(i) Show that the perpendicular height (w) of the slice shown is given by $w = \frac{h}{4} + 5$. 2

(ii) Find an expression for the volume of the slice shown in the diagram in terms of h. 2

(iii) Find the volume of the solid to 1decimal place. 2

QUESTION 1 (cont)

c) (i) Prove that the hyperbola with equation $x^2 - y^2 = a^2$ is the hyperbola $XY = \frac{1}{2}a^2$ referred to different axes. 2

(ii) Find the coordinates of the vertices, foci and the equations of the directrices of $XY = 4$ 2

QUESTION 2 Start a NEW PAGE (15 Marks)

(a) The normal at the point $P\left(cp, \frac{c}{p}\right)$ on the hyperbola $xy=c^2$, meets the x-axis at Q. M is the midpoint of PQ

(i) Show that the normal at P has the equation $p^3x - py = c(p^4 - 1)$. 2

(ii) Show that M has the coordinates $\left(\frac{c(2p^4-1)}{2p^3}, \frac{c}{2p}\right)$ 2

(iii) Hence or otherwise, find the equation of the locus of M. 2

(b) Using the hyperbola from part a) but where $p \neq \pm 1$.

(i) Write down the equation for the tangent at P. 1

(ii) If the tangent to the hyperbola at P meets the coordinate axes at A and B. Show that PA=PB. 2

(iii) Let the normal to the hyperbola at P meet the axes of symmetry of the hyperbola at C and D. Show that PC=PD=PA . 4

(iv) Sketch a graph of the hyperbola showing the results for parts so far. 1

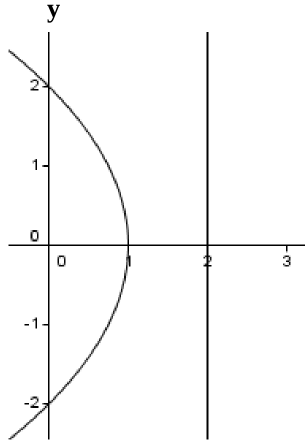
(v) Explain why ACBD is a cyclic quadrilateral and deduce that $BD \perp BC$ 1

(vi) Describe the geometry if $p = 1$ 1

Ext 2 Task 3 T2 2014

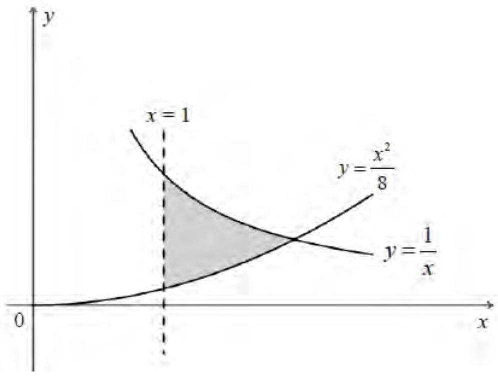
QUESTION 3 Start a NEW PAGE (15 Marks) **Marks**

a) A solid S is formed by rotating the region bounded by the parabola $y^2 = 4(1-x)$ and the y axis 360° about the line x = 2.



Find the volume of the solid S. 4

(b) The region bounded by $y = \frac{1}{x}$, $y = \frac{x^2}{8}$ and x=1 is rotated about the line x=1.



(i) Use the method of cylindrical shells to find an integral which gives the volume of the resulting solid of revolution 3

(ii) Find the volume of this solid of revolution 2

(c) A solid of mass 2kg is attached to an inextensible string of length 1•5 metres, the other end of the string being fixed. The mass rotates in a horizontal circle with an angular velocity of $\pi \ rad \ s^{-1}$, forming a conical pendulum. (Take $g = 10\,ms^{-2}$)

(i) Calculate the tension in the string. 3

(ii) Determine the angle between the string and the vertical axis. 1

(iii) Find the radius of the rotation. 1

(iv) What is the effect on the motion of the particle if the mass is doubled? 1

QUESTION 4 Start a NEW PAGE (15 Marks)

(a) A particle of mass 2kg is projected vertically upwards from a point A with velocity *u* m/s. It experiences a resistive force, in Newtons, of 10% of the square of its velocity *v* metres per second. The highest point reached is B directly above A. Assume g =10ms^{-2}, and take upwards as the positive direction

(i) Show that the acceleration of the particle as it rises is given by $\ddot{x} = -(\frac{v^2 + 200}{20})$ 1

(ii) Show that the distance x metres of the particle from A as it rises is given by 3

$$x = 10 \ln\left(\frac{200 + u^2}{200 + v^2}\right)$$

(iii) Show that the time t seconds that the particle takes to reach a velocity of v metres per second is given by $t=\sqrt{2}\left(\tan^{-1}\frac{u}{10\sqrt{2}} - \tan^{-1}\frac{v}{10\sqrt{2}}\right)$ 2

(iv) Now suppose we take two of the 2 kg particles described above. One of the particles is projected upwards from A with an initial velocity $10\sqrt{2}$ ms^{-1} then, $\frac{3\sqrt{2}}{5}$ seconds later the other particle is projected upwards from A with initial velocity $20\sqrt{2}$ ms^{-1}. Will the second particle catch up to the first particle before the first particle reaches its maximum height? You must explain your reasoning and show working. 3

(b) A particle is allowed to fall under gravity from rest in a medium which exerts a resistance proportional to the speed (v) of the particle.

(i) Show that the particle reaches a terminal velocity T given by 2

$T = \frac{g}{k}$ (where *k* is a positive constant).

(ii) Show that the distance fallen to reach half its terminal velocity 4

$\frac{T}{2}$ is given by $x = \frac{T^2}{g}\ln 2 - \frac{T^2}{2g}$.

END OF EXAMINATION

i) i) $\frac{x^2}{4}+y^2=1$ — 1 m — well done

ii) $\text{Area} = \frac{1}{2}\cdot 2|y|\cdot 2|y|\sin 60°$

$= 2y^2\cdot\frac{\sqrt{3}}{2}$

$= \sqrt{3}y^2$ — 1 m

When $x=k$ $\quad y^2 = 1-\frac{k^2}{4}$ — Some forgot $x=k$ $-\frac{1}{2}$ m

$\text{Area} = \sqrt{3}\left(1-\frac{k^2}{4}\right)$

$= \frac{\sqrt{3}}{4}(4-k^2)$ — 1 m

iii) $\Delta V = \frac{\sqrt{3}}{4}(4-k^2)\Delta k$

$V = \lim_{\Delta k\to 0}\sum_{k=-2}^{2}\frac{\sqrt{3}}{4}(4-k^2)\Delta k$

If use $V = 2\int_0^2 \frac{\sqrt{3}}{4}(4-k^2)\,dk$ must mention even function or symmetrical

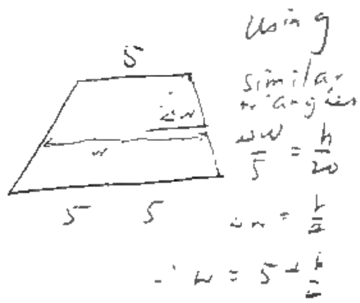
$= \int_{-2}^{2}\frac{\sqrt{3}}{4}(4-k^2)\,dk$ — 1 m

$= 2\int_0^2\frac{\sqrt{3}}{4}(4-k^2)\,dk$

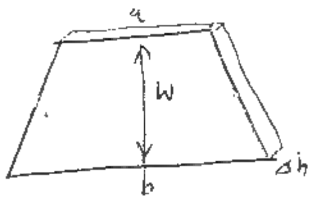
$= \frac{\sqrt{3}}{2}\left(4k-\frac{k^3}{3}\right)\Big]_0^2$

$= \frac{\sqrt{3}}{2}\left(8-\frac{8}{3}\right)$

$= \frac{8\sqrt{3}}{3}$ or $\frac{8}{\sqrt{3}}$ unit^3 — 1 m



b i)



$w(h)$ is a linear function

$w(h) = mh + c$ — 1 m

when $h=0$ $\quad w=5$ $\therefore c=5$

$h=20$ $\quad w=10$

$10 = 20m+5$, $m=\frac{1}{4}$

$\therefore w = \frac{h}{4}+5$ — 1 m

Some students assume symmetrical or right-angled trapezium -1 m

must mention similar triangles

Similarly

$a(h) = mh + b$

when $h = 0$ $\quad a = 0$ $\quad \therefore b = 0$

$h = 20$ $\quad a = 8$

$8 = 20m$ $\quad \therefore m = \frac{2}{5}$

$\therefore a = \frac{2h}{5}$

$b = \frac{2h}{5} + 4$ — 1 m

$\Delta V = \frac{1}{2}\left(\frac{h}{4} + 5\right)\left[\frac{2h}{5} + \frac{2h}{5} + 4\right]\Delta h$ — 1 m

$V = \lim\limits_{\Delta h \to 0} \sum\limits_{h=0}^{20} \frac{1}{2}\left(\frac{h}{4} + 5\right)\left(4 + \frac{4h}{5}\right)\Delta h$

$= \frac{1}{2}\int_0^{20} \left(\frac{h}{4} + 5\right)\left(4 + \frac{4h}{5}\right) dh$ — 1 m

$= \int_0^{20} \frac{h^2}{10} + \frac{h}{2} + 2h + 10 \ dh$

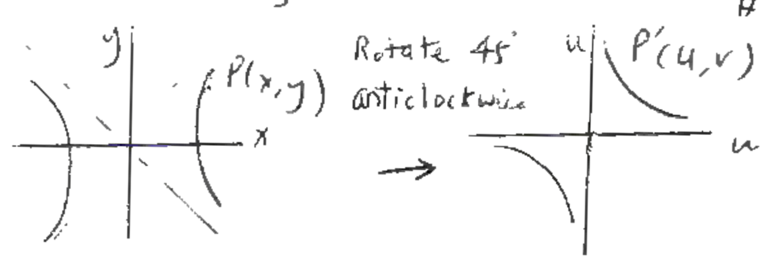
$= \left(\frac{h^3}{30} + \frac{5h^2}{4} + 10h\right)\Big]_0^{20}$

$= \frac{2900}{3} \text{ cm}^3$ or 966.7 cm^3 # — 1 m

Similarly

The front & rear sides may not be symmetrical

c i)



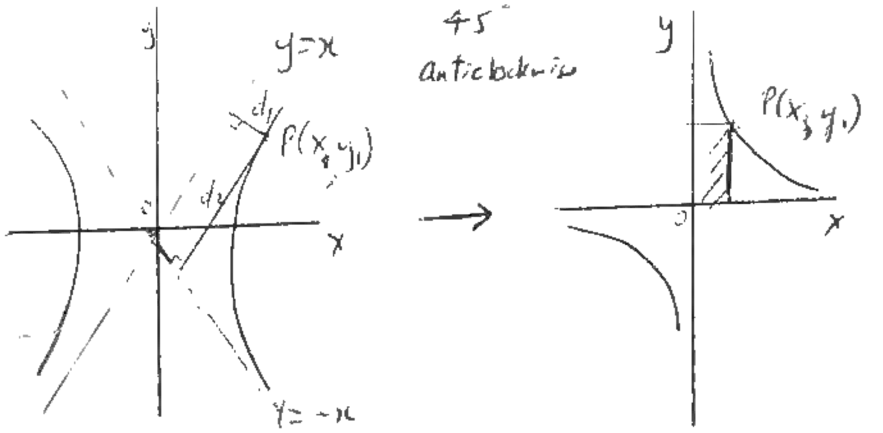
$P(x,y)$ represents $z = xi + y$, $P'(u,v)$ represents $(\text{cis}\, 45°)\cdot z$

$u + iv = \frac{1}{\sqrt{2}}(1+i)(x+iy) = \left(\frac{x-y}{\sqrt{2}}\right) + i\left(\frac{x+y}{\sqrt{2}}\right)$ — 1 m

$u = \frac{x-y}{\sqrt{2}}, \quad v = \frac{x+y}{\sqrt{2}}$

$uv = \frac{x^2 - y^2}{2} = \frac{a^2}{2}$ (since P lies on $x^2 - y^2 = a^2$) — 1 m

(i) $x^2 - y^2 = a^2$ Rotate 45° anticlockwise $xy = \frac{a^2}{2}$ $(= c^2)$



Using this method must mention equation of asymptotes otherwise −1 m.

$d_1 = \frac{|x_1 - y_1|}{\sqrt{2}}$ $\qquad$ $d_2 = \frac{|x_1 + y_1|}{\sqrt{2}}$ $\qquad$ 1m

Area of rectangle $= \frac{|x_1^2 - y_1^2|}{2} = \frac{a^2}{2}$ $\qquad$ 1m

(Since $P(x_1, y_1)$ lies on $x^2 - y^2 = a^2$)

After an anticlockwise rotation of 45°, area of rectangle is still $\frac{a^2}{2}$ (Now the x, y axes become the new asymptote)

$\therefore x_1 y_1 = \frac{a^2}{2}$

$\therefore$ the hyperbola with equation $x^2 - y^2 = a^2$ is the hyperbola $xy = \frac{a^2}{2}$ referred to different axes

(ii) Vertices $(2, 2)$, $(-2, -2)$

Foci $(2\sqrt{2}, 2\sqrt{2})$, $(-2\sqrt{2}, -2\sqrt{2})$

Directrices $x + y = \pm 2\sqrt{2}$

1 m for 1 correct
2 m for all 3 cor
no half mark.

MATHEMATICS Extension 2: Question 2

Suggested Solutions	Marks	Marker's Comments
(i) $xy = c^2 \quad \therefore y = c^2/x$		
$dy/dx = \frac{-c^2}{x^2}$		
when $x = cp$, $m = \frac{-c^2}{c^2p^2} = \frac{-1}{p^2}$		
$\therefore$ gradient of normal is p^2	1	generally very well done.
eqn of normal is: $y - \frac{c}{p} = p^2(x - cp)$		
$py - c = p^3x - cp^4$		
$cp^4 - c = p^3x - py$	1	
$c(p^4-1) = p^3x - py$		
(ii) when $y = 0$, $c(p^4-1) = p^3x$		
$\therefore x = \frac{c(p^4-1)}{p^3}$		
$Q\left(\frac{c(p^4-1)}{p^3}, 0\right)$	1	
Midpoint is $\left(\frac{\frac{c(p^4-1)}{p^3} + cp}{2}, \frac{0 + c/p}{2}\right)$		
$= \left(\frac{c(p^4-1) + cp^4}{2p^3}, \frac{c}{2p}\right)$	1	
$= \left(\frac{2cp^4 - c}{2p^3}, \frac{c}{2p}\right)$		
(iii) $X = \frac{2cp^4 - c}{2p^3} \qquad y = \frac{c}{2p}$		
$\therefore p = \frac{c}{2y}$		
sub into x coordinate		
$\therefore x = \frac{2cp^4 - c}{2p^3} = cp - \frac{c}{2p^3}$		
$\therefore x = c\left(\frac{c}{2y}\right) - \frac{c}{2\left(\frac{c^3}{8y^3}\right)}$	1	

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MATHEMATICS Extension 2: Question 2

Suggested Solutions	Marks	Marker's Comments
$\therefore x = \frac{c^2}{2y} - \frac{8y^3c}{2c^3}$		you had to substitute into x, and do at least 2 further lines of working to get ②
$\therefore x = \frac{c^2}{2y} - \frac{4y^3}{c^2}$	1	
or $xc^2 = \frac{c^4}{2y} - 4y^3$ or $2yxc^2 = c^4 - 8y^4$		
(b)(i) $x + p^2y = 2cp$	1	a lot of students wasted time by proving this formula
(ii) when $x=0$, $p^2y = 2cp$ $\therefore A(0, 2c/p)$ when $y=0$, $x=2cp$ $\therefore B(2cp, 0)$	1	needed to get both correct to get the 1 mark.
midpoint of AB is $\left(\frac{0+2cp}{2}, \frac{2c/p+0}{2}\right)$ $=\left(cp, \frac{2c}{2p}\right)$ $=\left(cp, c/p\right)$ $= P$ $\therefore P$ is the midpoint of AB.	1	some students did it by using the distance formula.
(iii) $y = x$ and $p^3x - py = c(p^4-1)$ $p^3x - px = c(p^4-1)$ $x(p^3-p) = c(p^4-1)$ $x = \frac{c(p^4-1)}{p^3-p}$ $= \frac{c(p^2-1)(p^2+1)}{p(p^2-1)}$ $= \frac{c(p^2+1)}{p}$ $\therefore C\left(\frac{c(p^2+1)}{p}, \frac{c(p^2+1)}{p}\right)$	1	Some students didn't simplify the algebra, and that made it harder for them in applying it.
$y = -x$ and $p^3x - py = c(p^4-1)$ $p^3x + px = c(p^4-1)$ $x(p^3+p) = c(p^4-1)$ $x = \frac{c(p^4-1)}{p(p^2+1)} = \frac{c(p^2-1)}{p}$		

$\therefore D\left(\frac{c(p^2-1)}{p}, \frac{-c(p^2-1)}{p}\right)$ 1

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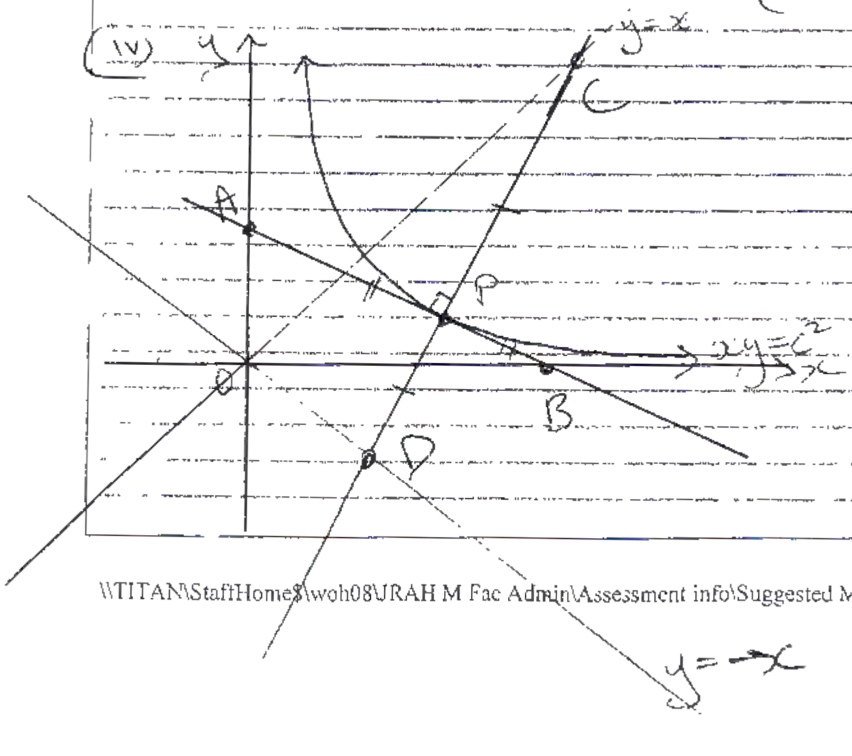
MATHEMATICS Extension 2: Question 2

Suggested Solutions	Marks	Marker's Comments
midpoint of CD is: $\left(\dfrac{\frac{c(p^2+1)}{p}+\frac{c(p^2-1)}{p}}{2}, \dfrac{\frac{c(p^2+1)}{p}-\frac{c(p^2-1)}{p}}{2}\right)$		
$=\left(\frac{2cp^2}{2p}, \frac{2c}{2p}\right)$		½ m
$=\left(cp, c/p\right)$		
$\therefore$ P is the midpoint of CD		
now $PC=\sqrt{\left(cp-\frac{c(p^2+1)}{p}\right)^2+\left(\frac{c}{p}-\frac{c(p^2+1)}{p}\right)^2}$		
$=\sqrt{\left(\frac{cp^2-cp^2-c}{p}\right)^2+\left(\frac{c-cp^2-c}{p}\right)^2}$		a lot of fudging going on
$=\sqrt{\frac{c^2}{p^2}+\frac{c^2p^4}{p^2}}$		
$=\frac{c}{p}\sqrt{1+p^2}$		
$PA=\sqrt{(cp-0)^2+\left(\frac{c}{p}-\frac{2c}{p}\right)^2}$		
$=\sqrt{c^2p^2+\frac{c^2}{p^2}}$		
$=\frac{c}{p}\sqrt{p^4+1}$		
$=PC$		
$\therefore PA=PC=PD$ (as P midpoint of CD)		
(iv) 		① needed to see A,B,C,D,P and $y=\pm x$ to get the 1

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MATHEMATICS Extension 2: Question 2

Suggested Solutions	Marks	Marker's Comments
(v) The diagonals AB and CD bisect each other at right angles, AB = CD so ACBD is a square. It is a cyclic quadrilateral as the opposite angles are supplementary. $BD \perp BC$ (angle B in a square is $90°$)		* some students forgot to prove why $B = 90°$
(vi) If $p = 1$, the normal becomes the axis of symmetry, ie $y = x$. So C has infinite values and D goes to the origin.	1	you needed all 3 pieces of information to get the mark. 2 out of 3 = 0 m

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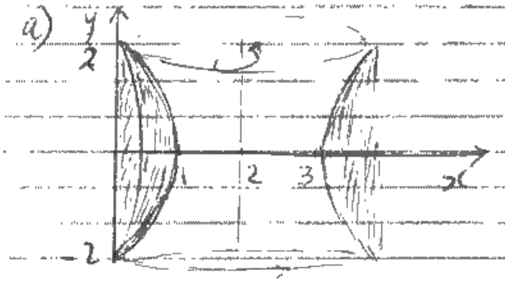
EXTENSION 2.

MATHEMATICS: Question…3…

Suggested Solutions	Marks	Marker's Comments

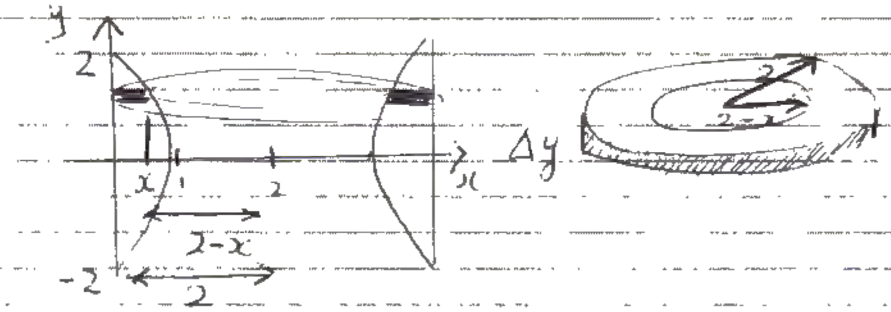
a)



$y^2 = 4(1-x)$

2 methods.

Method 1 SLICES.



$$y^2 = 4(1-x)$$

$$x = \frac{4-y^2}{4} = 1 - \frac{y^2}{4}$$

$$\Delta V \doteq \pi\left[2^2 - (2-x)^2\right]\Delta y$$

$$\doteq \pi\left[4 - 4 + 4x - x^2\right]\Delta y$$

$$\doteq \pi x\left[4-x\right]\Delta y$$

$$\doteq \pi\left[\frac{4-y^2}{4}\right]\left[4 - \frac{(4-y^2)}{4}\right]\Delta y$$

$$\doteq \pi\left[\frac{4-y^2}{4}\right]\left[\frac{12+y^2}{4}\right]\Delta y$$

$$\doteq \frac{\pi}{16}\left[48 - 8y^2 - y^4\right]\Delta y$$

$$V = \lim_{\Delta y \to 0}\sum_{y=-2}^{2}\frac{\pi}{16}\left[48 - 8y^2 - y^4\right]$$

$$= \frac{\pi}{16}\int_{-2}^{2}(48 - 8y^2 - y^4)\,dy$$

$$= \frac{2\pi}{16}\int_{0}^{2}(48 - 8y^2 - y^4)\,dy \quad \text{(even function)}$$

$$= \frac{\pi}{8}\left[48x - \frac{8y^3}{3} - \frac{y^5}{5}\right]_0^2$$

$$= \frac{128\pi}{15}$$

Volume is $\frac{128\pi}{15}$ units^3

Marks: 4

Marker's Comments:

4 for complete answer.

No marks deducted for missing explanation or working

1 ΔV statements

1 sub y into ΔV

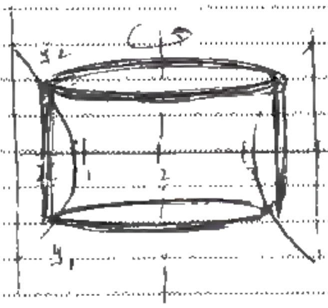
1 simplify algebra in integral

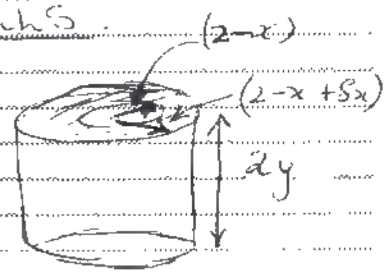
1 integrate and answer.

EXTENSION
MATHEMATICS: Question 3.

Suggested Solutions	Marks	Marker's Comments

Method 2 SHELLS

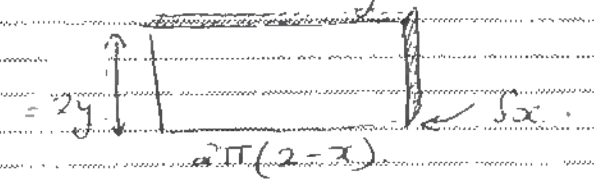




$\Delta V = \pi\left[(2-x+\delta x)^2 - (2-x)^2\right] \times 2y$

or unroll cylinder $\qquad y^2 = 4(1-x)$

$y = 2\sqrt{1-x} \quad y>0$



$\Delta V = 2\pi(2-x)\times 2y\,\delta x$

$= 4\pi(2-x)\times 2\sqrt{1-x}\,\delta x$

$V = \lim_{\delta x \to 0} 8\pi \sum_{x=0}^{1} (2-x)\sqrt{1-x}\,\delta x$

$= 8\pi\int_0^1 (2-x)\sqrt{1-x}\,dx$

Use substitution $u = 1-x \quad \frac{du}{dx} = -1$

$x = 1 \quad u = 0$

$x = 0 \quad u = 1$

$V = -8\pi\int_1^0 (1+u)\sqrt{u}\,du$

$= 8\pi\int_0^1 u^{\frac{1}{2}} + u^{\frac{3}{2}}\,du$

$= 8\pi\left[\frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2}\right]_0^1$

$= \frac{128\pi}{15}$

Volume is $\frac{128\pi}{15}$ units^3

Marker's Comments:

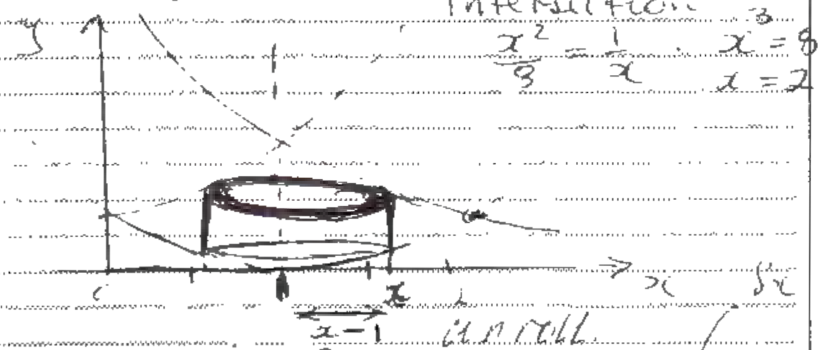
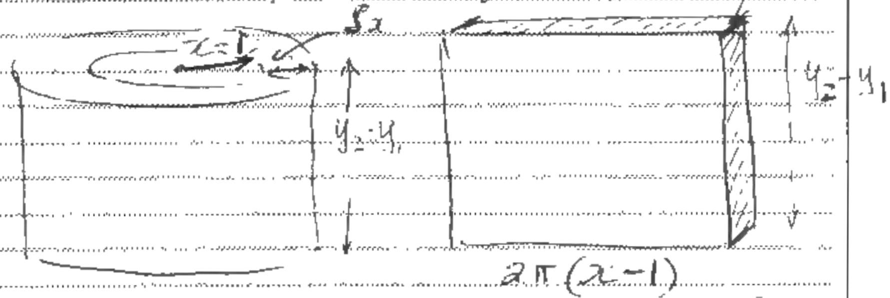
① ΔV statement

① sub x into ΔV to form integral

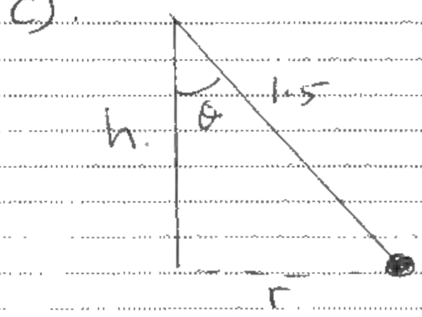
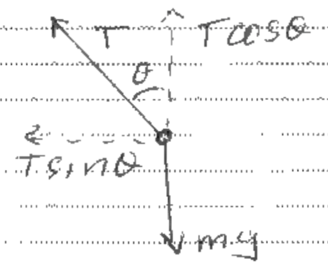
① complete integration

① complete answer.

EXTENSION
MATHEMATICS: Question 3

Suggested Solutions	Marks	Marker's Comments
(b)(i) $y = \frac{x^2}{8}$ $\quad y = \frac{1}{x}$. Point of intersection $\frac{x^2}{8} = \frac{1}{x}$. $x^3 = 8$, $x = 2$.	3	① point of intersection x value and limits on integral
		
		① diagrams showing variables and ΔV statement
$\Delta V = \pi\left[(x-1+\delta x)^2-(x-1)^2\right]\delta x$		
or $\Delta V = 2\pi(x-1)(y_2-y_1)\delta x$.		
$y_2 = \frac{1}{x} \quad y_1 = \frac{x^2}{8}$		
$\Delta V = 2\pi(x-1)\left[\frac{1}{x}-\frac{x^2}{8}\right]\delta x$.		
$V = \lim\limits_{\delta x\to 0} 2\pi\sum\limits_{x=1}^{2}(x-1)\left(\frac{1}{x}-\frac{x^2}{8}\right)\delta x$		① limit statement and correct integral (any form).
$V = 2\pi\int_1^2\left(1-\frac{1}{x}-\frac{x^3}{8}+\frac{x^2}{8}\right)dx$		
(ii) $V = 2\pi\left[x-\ln x-\frac{x^4}{32}+\frac{x^3}{24}\right]_1^2$		① integration
$= 2\pi\left\{\left[2-\ln 2-\frac{16}{32}+\frac{8}{24}\right]-\left[1-\ln 1-\frac{1}{32}+\frac{1}{24}\right]\right\}$		
$= 2\pi\left[\frac{79}{96}-\ln 2\right]$		① answer.
Volume $= \left[\frac{79}{48}-2\ln 2\right]\pi$ units^3		no cfe if answer does not contain log term.

EXTENSION 2
MATHEMATICS: Question 3

Suggested Solutions	Marks	Marker's Comments
c).  Force Diagram 	3	
(i) Horizontally $T\sin\theta = m\omega^2 r$ $m = 2 \quad \omega = \pi \quad g = 10$ $r = 1.5\sin\theta$ $T\cancel{\sin\theta} = 2\times\pi^2\times 1.5\cancel{\sin\theta}$ $T = 3\pi^2$ Tension $= 3\pi^2$ Newton.		① Horizontal Equation ① $r = 1.5\sin\theta$ ① Answer.
(ii) Vertically $T\cos\theta = mg$ $3\pi^2\cos\theta = 2\times 10$ $\cos\theta = \frac{20}{3\pi^2}$ $\theta = \cos^{-1}\left(\frac{20}{3\pi^2}\right) \approx 47^\circ 31'$	1	① correct answer. (exact form accepted)
(iii) $r = 1.5\sin\theta$ $= 1.5\sin\left[\cos^{-1}\left(\frac{20}{3\pi^2}\right)\right]$ ≈ 1.11 radius $\doteqdot 1.11$ m Alternatively $r = \frac{1}{2\pi^2}\sqrt{9\pi^4 - 400}$	1	① correct answer (Exact form accepted)
(iv) $T\sin\theta = m\omega^2 r = m\omega^2 l\sin\theta$ $T\cos\theta = mg$ $\cos\theta = \frac{mg}{m\omega^2 l} = \frac{g}{\pi^2 l}$. $\therefore \cos\theta$ is independent of m. $\therefore$ no change to motion for $\omega = \pi$	1	① no change to motion (no justification needed).

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a) i) x, v ; $x=0$; FORCES $mg + 0{\cdot}1v^2 = 20 + 0{\cdot}1v^2$ with given values $m\ddot{x} = \sum F_{up}$ (Newton's 2nd Law) $2\ddot{x} = -(20 + 0{\cdot}1v^2)$ $\ddot{x} = -\dfrac{(20+0{\cdot}1v^2)}{2}$ $\ddot{x} = -\left(\dfrac{200+v^2}{20}\right)$	1	Given result. Show needed either a clear diagram or words "Newton's 2nd Law" to explain the equation of motion.
ii) $\ddot{x} = v\dfrac{dv}{dx} = -\left(\dfrac{v^2+200}{20}\right)$	1	
$\displaystyle\int_u^v \frac{v\,dv}{v^2+200} = -\int_0^x \frac{dx}{20}$ $\left[\frac{1}{2}\ln(v^2+200)\right]_u^v = -\left[\frac{x}{20}\right]_0^x$	1	Most people scored full marks here. A few used $\frac{d}{dx}\left(\frac{v^2}{2}\right)$ and got into trouble with limits. There was a temptation to jump to the given result.
$\ln(v^2+200) - \ln(u^2+200) = -x/10$ $x = 10\left(\ln(u^2+200) - \ln(v^2+200)\right)$ $= 10\ln\left(\dfrac{200+u^2}{200+v^2}\right)$	1	
iii) $\ddot{x} = \dfrac{dv}{dt} = -\left(\dfrac{v^2+200}{20}\right)$ $\displaystyle\int_u^v \frac{dv}{v^2+200} = -\int_0^t \frac{dt}{20}$		

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$\left[\frac{1}{\sqrt{200}}\tan^{-1}\frac{v}{\sqrt{200}}\right]_u^v = -\left[\frac{t}{20}\right]_0^t$	1	
$\frac{1}{10\sqrt{2}}\left(\tan^{-1}\frac{v}{10\sqrt{2}} - \tan^{-1}\frac{u}{10\sqrt{2}}\right) = -\frac{t}{20}$		Nearly all were able to complete this
$\therefore t = \frac{20}{10\sqrt{2}}\left(\tan^{-1}\frac{u}{10\sqrt{2}} - \tan^{-1}\frac{v}{10\sqrt{2}}\right)$		
$\underline{\underline{t = \sqrt{2}\left(\tan^{-1}\frac{u}{10\sqrt{2}} - \tan^{-1}\frac{v}{10\sqrt{2}}\right)}}$	1	
iv) The first particle reaches a max ht. of $x_{max} = 10\ln\left(\frac{200+(10\sqrt{2})^2}{200}\right) = \underline{\underline{10\ln 2}}$ (m)		First mark for getting both of these values
after a time given by $t = \sqrt{2}\left(\tan^{-1}\frac{10\sqrt{2}}{10\sqrt{2}} - \tan^{-1}0\right)$ $= \sqrt{2}\tan^{-1}(1) = \underline{\underline{\frac{\sqrt{2}\pi}{4}}}$ (secs)	1	
For second particle there are two approaches:		
Ⓐ Find where 2nd particle is when $t = \frac{\pi\sqrt{2}}{4} - \frac{3\sqrt{2}}{5}$ $t \to v \to x$. Compare heights at this time		
Ⓑ Find time taken to reach $x = 10\ln 2$ $x \to v \to t$. Compare times taken (include offset)		

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(A) Find v when $t = \frac{\pi\sqrt{2}}{4} - \frac{3\sqrt{2}}{5}$ $\therefore \frac{\pi\sqrt{2}}{4} - \frac{3\sqrt{2}}{5} = \sqrt{2}\left(\tan^{-1}\frac{20\sqrt{2}}{10\sqrt{2}} - \tan^{-1}\frac{v}{10\sqrt{2}}\right)$ $\tan^{-1}\left(\frac{v}{10\sqrt{2}}\right) = \tan^{-1}(2) + \frac{3}{5} - \frac{\pi}{4}$ $= 0.92175\ldots$ $\therefore v = 10\sqrt{2}\tan(0.92175\ldots)$ $= 18.63996\ldots$	1	Rather too many people used their calculator in Degrees mode - not a good idea.
Thence $x = 10\ln\left(\frac{200+(20\sqrt{2})^2}{200+(18.6399)^2}\right)$ $= 6.025$ (3DP) $< 10\ln 2$ $(= 6.931)$ $\therefore$ Particle does not pass first in this time range.	1	
(B) Find v when $x = 10\ln 2$ $10\ln 2 = 10\ln\left(\frac{200+(20\sqrt{2})^2}{200+v^2}\right)$ $2 = \frac{1000}{200+v^2}$ $v^2 = 300$, $v = 10\sqrt{3}$ (up)	1	
Time since takeoff, T, given by $T = \sqrt{2}\left(\tan^{-1}\left(\frac{20\sqrt{2}}{10\sqrt{2}}\right) - \tan^{-1}\left(\frac{10\sqrt{3}}{10\sqrt{2}}\right)\right)$ $= 0.31264\ldots$ Time since $t=0$ is $T + 3\sqrt{2}/5$ $= 1.16117\ldots$ $> \frac{\pi\sqrt{2}}{4}$ $(= 1.1107\ldots)$ $\therefore$ Particle 2 does not catch first before top point	1	

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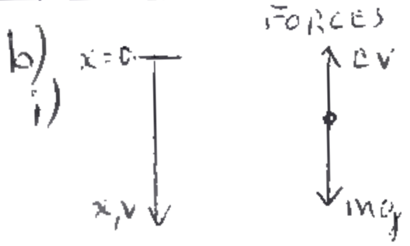
b) i)  $c>0$ since resistance opposes motion.		Should be an easy 2 marks but there was much fudging of m.
$m\ddot{x} = mg - cv$ $\therefore \ddot{x} = g - kv$ where $k = c/m > 0$	1	
as $\ddot{x} \to 0$, $v \to T$ $\therefore g - kT = 0$ $T = g/k$	1	
ii) $\therefore v\dfrac{dv}{dx} = g - kv$ $\displaystyle\int_0^{T/2} \frac{v\,dv}{g-kv} = \int_0^x dx$ $\displaystyle\frac{1}{k}\int_0^{T/2} \frac{kv-g+g}{g-kv}\,dv = \int_0^x dx$		
$\displaystyle\frac{1}{k}\int_0^{T/2}\left(\frac{g}{g-kv} - 1\right)dv = x$	1	
$\therefore \left[-\dfrac{1}{k^2}\ln\lvert g-kv\rvert - \dfrac{v}{k}\right]_0^{T/2} = x$		One mark for dealing adequatly with absolute value signs.
$0 < v < g/k$ so abs. values not nec.	1	
$\displaystyle -\frac{1}{k^2}\ln\left(\frac{g-kT/2}{g}\right) - \frac{T}{2k} = x$	1	Total omission of them, without explanation, lost one mark.
But $k = g/T$ from (1) $\therefore x = -\dfrac{T^2}{g^2}\ln\left(\dfrac{g-g/2}{g}\right) - \dfrac{T^2}{2g}$		
$= \dfrac{T^2}{g^2}\ln 2 - \dfrac{T^2}{2g}$	1	Given result, g need an extra line before end for ln 2